On the number of components of (k, g)-cages after vertex deletion

Yuqing Lin
School of Electrical Engineering and Computer Science
The University of Newcastle, NSW2308, Australia
e-mail: yuqing.lin@newcastle.edu.au

Camino Balbuena
Department de Matemàtica Aplicada III
Universitat Politècnica de Catalunya
e-mail: m.camino.balbuena@upc.edu

Mirka Miller
School of Information Technology and Mathematical Sciences
University of Ballarat, Ballarat, Victoria 3353, Australia
e-mail: m.miller@ballarat.edu.au

Abstract

A (k,g)-cage is a k-regular graph of girth g and with the least possible number of vertices. In this paper, we investigate the problem of how many connected components there will be after removing a cutset of up to k vertices from a (k, g)-cage.

1 Definitions

Throughout this paper, only undirected simple graphs without loops or multiple edges are considered. Unless otherwise stated, we follow [3] for terminology and definitions.

The vertex set (respectively, edge set) of a graph G is denoted by $V(G)$ (respectively, $E(G)$). If V' is a nonempty subset of V then the subgraph induced by V' is denoted by $G[V']$. Similarly, if E' is a nonempty subset of E then the subgraph induced by E' is denoted by $G[E']$. The subgraph obtained from G by deleting the vertices in V', together with their incident edges, is denoted by $G - V'$. The graph obtained from G by deleting a set of edges E' is denoted by $G - E'$.

The set of vertices adjacent to a vertex v is denoted by $N(v)$. If X is a nonempty subset of vertices, then $N(X)$ stands for the set $\left(\bigcup_{x\in X} N(x)\right) \setminus X$. The *degree* of a vertex v is $deg(v) = |N(v)|$, and a graph is called *regular* when all the vertices have the same degree. The *minimum degree* of the graph is denoted by δ. The *distance* $d(u, v)$ of two vertices u and v in $V(G)$ is the length of the shortest path between u and v. We also use the notion of a distance between a vertex v and a set of vertices X, written $d(v, X)$: it is the distance from v to a closest vertex in X.

The length of a shortest cycle in a graph G is called the *girth* of G. A k-regular graph with girth g is called a *(k, g)-graph.* A (k, g)-graph is called a *(k, g)-cage* if it has the least possible number of vertices. Throughout this paper, $f(k, g)$ stands for the order of a (k, g)-cage.

A graph G is *connected* if there is a path between any two vertices of G. A connected component of G is a maximal connected subgraph. Two vertices are in the same connected component if and only if there exists a path between them. A nonempty connected graph has one connected component.

Suppose that G is a connected graph. We say that G is *t-connected* if the deletion of at least t vertices of G is required to disconnect the graph. Similarly, we say that a graph is *t-edge-connected* if the deletion of at least t edges of G is required to disconnect the graph. A *vertex-cut* (respectively, *edge-cut*) of a graph is a set of vertices (respectively, edges), whose removal disconnects the graph. A graph is *maximally connected* (respectively, *maximally edge-connected*) if the minimum cardinality of vertex-cut (respectively, edge-cut) is equal to the minimum degree of the graph. An edge-cut W is called *trivial* if it contains all the edges incident with some vertex, that is, $\{xx_i \in E(G) : x_i \in N(x)\} \subseteq W$, for some $x \in V(G)$. A vertex-cut S is called *trivial* if there exists some vertex v in $G-S$ such that $N(v)$ is contained in S. A maximally edge-connected graph is *edge superconnected* if all its minimal edge-cuts (with cardinality equal to δ) are trivial. A *vertex superconnected* graph is defined similarly.

2 Introduction

Network reliability concerns the capability of an interconnection network to provide enough communication volume to support message exchanges, it is a very important parameter to study. It is customary to model a network structure as a graph and study the properties of network structures in graph theory terms, there are many parameters have been introduced to measure the reliability of a network structure. In a network with unreliable components such as wires or communication devices, disconnection is a major concern, which is caused by faulty nodes and/or faulty edges, preventing the disconnected parties from communicating with each other. The connectivity is used to measure the connectedness of a graph, i.e. how likely a graph is to be disconnected. In this paper, we shall study the connectivity of an important graph structure which is known as cages.

Cages were introduced by Tutte in [16]. For example, the cycle C_n is the unique $(2, n)$-cage, the complete graph K_n is the unique $(n-1, 3)$-cage, the complete bipartite graph $K_{n,n}$ is the $(n, 4)$-cage, the Petersen graph is the $(3, 5)$-cage, the Hoffman-Singleton graph is the $(7, 5)$-cage, and the Heawood graph is the $(3, 6)$-cage. Note that cages are not necessarily unique for each given pair of values k and g. For instance, there exist 18 non-isomorphic $(3, 9)$-cages, all of order 58 [2].

Much research has been carried out on cages; however, in general, not a lot is known about this structure. The study of the connectivity of cages has been suggested by several authors. In particular, in [4, 8] was proved that every (k, g)-cage with $k \geq 3$ is 3-connected. Furthermore, Fu, Huang, and Rodger [7] have proposed the following conjecture.

Conjecture 1 [7] *Every (k, g)-cage is maximally connected.*

There are many results available related to this conjecture, see [9, 10, 11, 12], but currently the conjecture is still open.

As we know that connectivity is a very rough measure of the vulnerability, many refinements of the classical connectivity concept have been introduced, for example, the toughness, the restricted connectivity, etc. In this paper, we investigate the number of connected components obtained from a (k, g)-cage by removing the vertices of a vertex-cut S with cardinality $|S| \leq k$. The results reveal new structural properties of (k, g)-cages.

3 New results

Jiang and Mubayi [8] proved the following theorem.

Theorem 3.1 *[8] Let S be a vertex-cut of a (k, g)-cage with $k \geq 3$ and $g \geq 5$. Then, the diameter of $G[S]$ is at least $\lfloor g/2 \rfloor$. Furthermore, the inequality is strict if $d_{G[S]}(u, v)$ is maximized for exactly one pair of vertices.*

Let $w(G-S)$ denote the number of components of $G-S$ where S is any vertex-cut. Using Theorem 3.1 it was shown in [13] the following result for cubic cages.

Theorem 3.2 *[13] Every cubic cage is quasi 4-connected, that is to say, any minimum vertex-cut of a cubic cage is the neighborhood of a vertex, say $N(v)$, and $w(G - N(v)) = 2$.*

Following the same line of reasoning, using Theorem 3.1 we immediately obtain the following corollary.

Corollary 3.1 *Let G be a (k,g)-cage with $k \geq 3$ and $g \geq 6$. For every vertex v of G, we have $w(G - N(v)) = 2$.*

Proof. Suppose G has a vertex v such that $G - N(v)$ has at least three components, one of them consisting of vertex v. Then $S = \{v\} \cup N(v)$ is a vertex-cut because $G - S$ has at least two or more components. As its induced subgraph $G[\{v\} \cup N(v)]$ has diameter 2, from Theorem 3.1, it follows that $2 \geq \lfloor g/2 \rfloor$, which is a contradiction because $g \geq 6$. ■

Next question concerns a nontrivial vertex-cut. Suppose G is an edge superconnected k-regular graph, and S is a nontrivial vertex-cut. The following theorem provides an upper bound for $w(G - S)$.

Theorem 3.3 *Let G be a k-regular edge superconnected graph. Then for any vertex-cut S the following statements hold:*

(i) If S is nontrivial then the number of components is $w(G - S) \leq k|S|/(k+1)$.

(ii) If the girth $g \geq 5$, $|S| \leq k$ and $k \geq 4$ is even, then either $w(G - S) \leq k - 2$ or G has vertex connectivity at most $k - 2$ except for $k = 4$ and S trivial, in which case $w(G - S) \leq 3$.

Proof. Let S be a nontrivial vertex-cut of G and let C_i $(i = 1, 2, \ldots, w(G - S))$ denote the components of the graph obtained when the vertices of S are removed from G. Let us also denote by $[S, V(C_i)]$ the set of edges joining a vertex in S with a vertex in C_i. Since S is nontrivial then every C_i has at least two vertices. Furthermore, since G is edge superconnected we have

$$\Big|[S, V(C_i)]\Big| \geq k + 1, \; i = 1, 2, \ldots, w(G - S). \tag{1}$$

Moreover, as the graph is k-regular, by (1) we have

$$w(G - S)(k + 1) \leq \sum_{i=1}^{w(G-S)} \Big|[S, V(C_i)]\Big| \leq k|S|. \tag{2}$$

Therefore $w(G-S) \leq k|S|/(k+1)$, hence item (i) holds. To prove (ii) assume $k \geq 4$ even and $|S| \leq k$. First suppose that S is nontrivial, thus we may apply item (i) whence $w(G-S) \leq k-1$. Suppose $w(G-S) = k-1$ because otherwise item (ii) is valid. Substituting this value in (2) and taking into account (1) it follows that there exists a component $\widehat{C}$ such that

$$\left|[S, V(\widehat{C})]\right| = k+1. \tag{3}$$

Then $|N(S) \cap V(\widehat{C})| \leq \left|[S, V(\widehat{C})]\right| = k+1$. If $|N(S) \cap V(\widehat{C})| = k+1$ then each vertex of $N(S) \cap V(\widehat{C})$ is adjacent to exactly one vertex of S, which means that each of these $k+1$ vertices has degree $k-1$ in $\widehat{C}$. In other words, each vertex of $\widehat{C}$ has degree k except $k+1$ (which is odd) vertices which have odd degree $k-1$, which is impossible. If $|N(S) \cap V(\widehat{C})| = k$ then, by (3), one vertex of $N(S) \cap V(\widehat{C})$ has degree $k-2$ in $\widehat{C}$, while the others must have degree $k-1$ in $\widehat{C}$. That is, $\widehat{C}$ has $k-1$ vertices of odd degree $k-1$, which is again impossible. If $|N(S) \cap V(\widehat{C})| = k-1$ then we would obtain that there is an odd number of vertices in $N(S) \cap V(\widehat{C})$ of odd degree in $\widehat{C}$ (either $k-2$ vertices of degree $k-1$ and 1 vertex of degree $k-3$, or $k-3$ vertices of degree $k-1$), which is also impossible. Thus we have $|N(S) \cap V(\widehat{C})| \leq k-2$. Now, in order to prove that $N(S) \cap V(\widehat{C})$ is a vertex-cut of G, it is enough to prove $V(\widehat{C}) \neq N(S) \cap V(\widehat{C})$. Otherwise, we would have $|V(\widehat{C})| + |S| = |N(S) \cap V(\widehat{C})| + |S| \leq 2k-2$. As S is nontrivial every vertex $v \in N(S) \cap V(\widehat{C})$ has a neighbor $w \in N(S) \cap V(\widehat{C})$. Since the girth $g \geq 5$ then $N(v) \cap N(w) = \emptyset$, hence $|V(\widehat{C})| + |S| \geq |\{u, v\}| + |N(u) - w| + |N(w) - u| \geq 2 + 2(k-1) = 2k$, which is a contradiction. Then G has vertex connectivity at most $k-2$ if S is a nontrivial vertex-cut. To finish the proof, assume S is trivial, which means that there exists a vertex v in $G-S$ such that $S = N(v)$ because $|S| \leq k$. That is, there is one component formed by vertex v and the remaining components must have cardinality at least 2, because the girth $g \geq 5$. Therefore, (1) still holds for every component except for that formed by vertex v. Hence we have

$$(w(G-N(v)) - 1)(k+1) \leq \sum_{C_i, v \notin V(C_i)} \left|[N(v), V(C_i)]\right| \leq k(k-1), \tag{4}$$

which implies $w(G-N(v)) \leq k-1$. Suppose that $w(G-N(v)) = k-1$ and $k \geq 6$, from (1) and (4) we obtain again (3), so that the reasoning is the same as above to prove that the vertex connectivity is at most $k-2$. ■

Combining results in [11] and [10], we have concluded in [10] that:

Theorem 3.4 [10] All (k,g)-cages are edge superconnected, that is, all the minimal edge-cuts are trivial.

Therefore, combining Theorems 3.3 and 3.4 and Corollary 3.1, we obtain the following result, whose proof is immediate and, therefore, is omitted.

Corollary 3.2 *Let G be a (k,g)-cage with $g \geq 5$. Then for any vertex-cut S the following statements hold:*

(i) If S is nontrivial then $|S|/w(G-S) \geq (k+1)/k$.

(ii) If the girth $g \geq 5$, S is nontrivial with $|S| \leq k$ and $k \geq 4$ is even, then either $w(G-S) \leq k-2$ or G has vertex connectivity at most $k-2$.

(iii) If S is trivial with $|S| \leq k$, then $w(G-S) = 2$.

The *toughness* $\tau(G)$ of a non complete graph G is defined as $\tau(G) = \min\{|S|/w(G-S)\}$, where the minimum is taken over all cut-sets S. Thus, Corollary 3.2 is a first step to find a lower bound for the toughness of a (k,g)-cage with $k \geq 3$ and $g \geq 5$. It remains to find a lower bound for $|S|/w(G-S)$ for every trivial vertex-cut S with $|S| \geq k+1$.

Furthermore, in [12] it was proved the following result.

Theorem 3.5 *[12] Every (k,g)-cage with $k \geq 4$ and $g \geq 10$ is 4-connected.*

As an immediate consequence of Theorems 3.4 and 3.5 and Corollary 3.2 the following result is derived for $(4,g)$-cages.

Corollary 3.3 *Let G be a $(4, g)$-cage with $g \geq 10$. Then either G is vertex superconnected or $w(G - S) = 2$, for every nontrivial vertex-cut S with $|S| = 4$.*

Since Conjecture 1 is still open, to reveal more structural properties of (k, g)-cages, next, we would like to show that the deletion of at most $k - 1$ vertices from a (k, g)-cage G, with $k \geq 4$ and $g \geq 11$, results in a new graph with at most 2 connected components. Hopefully, this result will contribute towards settling Conjecture 1.

In order to prove this result, we shall often use the following well known monotonicity theorem.

Theorem 3.6 *[5, 7] If $k \geq 3$ and $3 \leq g_1 < g_2$ then $f(k, g_1) < f(k, g_2)$.*

Moreover, we need the following lemma which has been proved in [1, 6, 14, 15].

Lemma 3.1 *[1, 6, 14, 15] Let G be a graph with girth g, and minimum degree δ. Assume that S is a vertex-cut of cardinality $|S| \leq \delta - 1$. Then, for any connected component C in $G - S$, there exists some vertex $x \in V(C)$ such that $d(x, S) \geq \lfloor (g - 1)/2 \rfloor$.*

We begin by proving a technical lemma.

Lemma 3.2 *Let G be a k-regular graph and assume that $S = \{s_i : i = 1, \ldots, |S|\}$ is a vertex-cut. Suppose that A and B are two connected components of $G - S$ and $C = G - S - (A \cup B)$. Then one of the following numbers is even*

(i) $k|S| - \sum_{j=1}^{|S|} |N_A(s_i)|$,

(ii) $k|S| - \sum_{j=1}^{|S|} |N_B(s_i)|$,

(iii) $k|S| - \sum_{j=1}^{|S|} |N_C(s_i)|$.

Proof. Denote by $G[S]$ the graph induced by S, and by $E(G[S])$ the set of edges in the induced graph $G[S]$. Clearly, the total degree sum of all the vertices in $G[S]$ is $2|E(G[S])|$. Then we have

$$k|S| = 2|E(G[S])| + \sum_{i=1}^{|S|} |N_A(s_i)| + \sum_{i=1}^{|S|} |N_B(s_i)| + \sum_{i=1}^{|S|} |N_C(s_i)|.$$

Multiplying this equality by three, we can write

$$(k|S| - \textstyle\sum_{i=1}^{|S|} |N_A(s_i)|) + (k|S| - \sum_{i=1}^{|S|} |N_B(s_i)|) + (k|S| - \sum_{i=1}^{|S|} |N_C(s_i)|)$$

$$= 2(3|E(G[S])| + \textstyle\sum_{i=1}^{|S|} |N_A(s_i)| + \sum_{i=1}^{|S|} |N_B(s_i)| + \sum_{i=1}^{|S|} |N_C(s_i)|).$$

Therefore, one of the three summands on the left hand side of this equality must be even. Hence the result holds. ■

In the proof of the next result we use the following notation. By $N_G^2(v)$ we mean the vertices w in the graph G which are at distance 2 from vertex v.

Theorem 3.7 *Let G be a (k,g)-cage with $g \geq 11$. Then either G is maximally connected or $w(G-S) = 2$, for every vertex-cut S of G of cardinality $|S| < k$.*

Proof. Let S be a vertex-cut of G with $|S| < k$. Hence by the results proved in [7, 4] and Theorem 3.5 we know $k \geq 5$. Assume that $G - S$ contains more than 2 connected components, say A and B, and a further (not necessary connected) component C. Since $|S| < k$, by Lemma 3.1, it is clear that in A (respectively, B and C), there exists a vertex u (respectively, v and w), which has distance $(g-1)/2$ (for g odd) or $g/2 - 1$ (for g even) to S.

Denote the neighbors of u by u_i, for $i = 1, 2, \ldots, k$, so $N(u) = \{u_1, u_2, \ldots, u_k\}$. Similarly, denote the neighbors of u_i by u_{ij}, for $j = 1, 2, \ldots, k-1$, so $N(u_i) = \{u, u_{i1}, u_{i2}, \ldots, u_{ik-1}\}$. We shall use the same notation for the neighbors of v and w. We also denote the vertices in S by s_i, $i = 1, 2, \ldots, |S|$. For each s_i, denote the set of neighbors of s_i in A by A_i. Similarly, we define the sets B_i and C_i.

Without loss generality, we may assume, by Lemma 3.2, that $k|S| - \sum_{i=1}^{|S|} |A_i|$ is even. Additionally, we assume that $|B| \leq |C|$. Let us consider the following three subgraphs:

$H = G[(A \cup S) - (\{u\} \cup N(u) \cup N(u_k))]$,

$K = G[B \cup S]$, and

$K' = G[(B \cup S) - (\{v\} \cup N(v) \cup N(v_k))]$.

We will construct a k-regular graph G^*, with girth at least g, by removing the edges between vertices in S, if any, and joining the vertices of H, K and K' by some new edges. The order of the resulting graph will be

$$|V(G^*)| = |V(H)| + |V(K)| + |V(K')| = |V(A)| + 2|V(B)| - 4k + 3|S| \leq |V(G)| - k - 3.$$

Sin $k \geq 5$ we will have constructed a (k, g)-graph with fewer vertices than the number of vertices of the original graph G and, since G was assumed to be a (k, g)-cage, this is a contradiction, by Theorem 3.6.

Next, we describe the new edges connecting vertices of H, K and K'. For easy presentation, we shall still make reference to the vertex u_i or v_i, even though they do not exist in these subgraphs.

A vertex s_i in subgraph H will be arbitrarily matched with a vertex v_i in K. We shall remove $k - |A_i|$ edges between v_i and its neighbors other than v, and connect these neighbors of degree $k-1$ to s_i. It is clear that after these operation, vertex s_i in subgraph H has degree k, and v_i in K has degree $|A_i|$, which are shown as the vertices on the bottom in the graph depicted in Figure 1.

The vertex s_i in K will be arbitrarily matched with a vertex v_i in K' and connected to $k - |B_i|$ vertices v_{ij} in $N(v_i)$ in K'. It is clear that after this operation, the vertices s_i and v_{ij} will have degree k. We shall make the same connections between s_i in K' and $k - |B_i|$ vertices u_{ij} in $N(u_i)$ in H.

Now, let u fix the degrees of the vertices v_i in K. The total current degree of these vertices is

$$\sum_{i=1}^{|S|} |A_i| + (k - |S|)\, k = k^2 - (k|S| - \sum_{i=1}^{|S|} |A_i|).$$

In other words, we still need to find $k|S| - \sum_{i=1}^{|S|} |A_i|$ vertices in the graph to connect to the vertices v_i in K. Based on our assumption, we know that $k|S| - \sum_{i=1}^{|S|} |A_i|$ is even. We shall connect vertex v_i to some vertices of $N(v_{ki})$ in K' and to some vertices of $N(u_{ki})$ in H. More precisely, for each v_i, we shall connect v_i to $\lceil (k - |A_i|)/2 \rceil$ vertices in $N(v_{ki})$ and to $\lfloor (k - |A_i|)/2 \rfloor$ vertices in $N(u_{ki})$, or vice versa, in such a way, that we will use $(k|S| - \sum_{i=1}^{|S|} |A_i|)/2$ vertices from $N^2_H(u_k))$ and $(k|S| - \sum_{i=1}^{|S|} |A_i|)/2$ vertices from $N^2_{K'}(u_k))$ to connect to $N_K(v)$. This implies that the number of leftover vertices of degree $k-1$ in subgraphs H and K' is the same. Then we shall pair these vertices and connect them by an edge. The resulting graph G^* is shown in Figure 1.

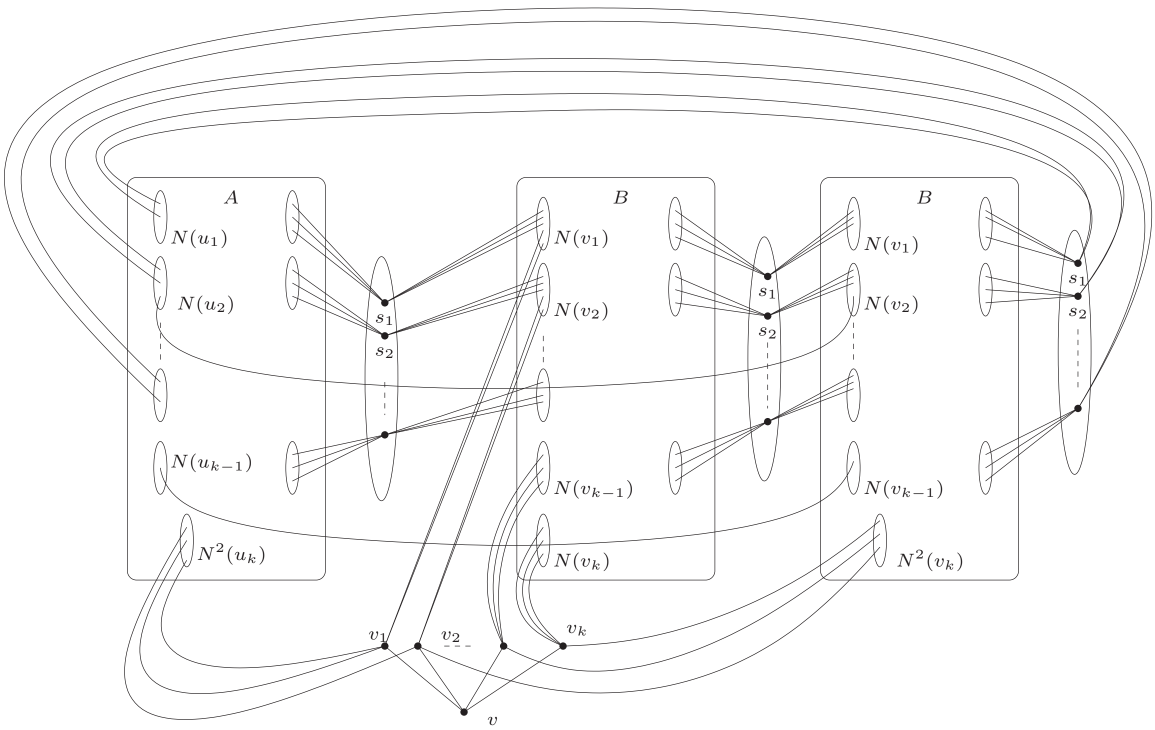

Figure 1: Structure of G^*.

To finish the proof, it is enough to show that there are no small cycles in G^*. Considering the new added edges, clearly, any new cycle, say $\mathcal{C}$, which was introduced in the construction has to utilize at least two new edges.

Case 1: The cycle $\mathcal{C}$ goes through two new edges incident with a common vertex $s_i \in H$. Then the cycle $\mathcal{C}$ must go through two distinct vertices v_{ij} in $N(v) \subset V(K)$ at distance $g-2$ from each other, because the edges $v_i v_{ij}$ have been deleted from K. Therefore, the cycle $\mathcal{C}$ has length at least g. A similar reasoning applies if the cycle $\mathcal{C}$ goes through two new edges incident with a common vertex $s_i \in K \cup K'$.

Case 2: The cycle $\mathcal{C}$ goes through two new edges incident with a common vertex $v_i \in K$. The reasoning is the same as in Case 1 if the cycle goes through two distinct vertices v_{kij} in $N(v_{ki}) \subset V(K')$ or two distinct vertices u_{kij} in $N(u_{ki}) \subset V(H)$. So assume that the cycle goes through one vertex v_{kij} in $N(v_{ki}) \subset V(K')$ and the other u_{kij} in $N(u_{ki}) \subset V(H)$. Since u_{ij} in H has been connected with s_i in K', we have, by Lemma 3.1, that

$$d_{G^*}(u_{kij}, v_{kij}) = d_H(u_{kij}, u_{ij}) + 1 + d_{K'}(s_i, v_{kij}) \geq g - 5 + 1 + \lfloor (g-1)/2 \rfloor - 3 \geq g - 2$$

because $g \geq 11$.

Case 3: The cycle $\mathcal{C}$ goes through two new edges incident with two distinct vertices s_i in H. Then the cycle $\mathcal{C}$ must go through two distinct vertices in $N^2(v) \subset V(K)$, which have distance at least $g-4$ from each other. As the vertices in S have distance at least 2 from each other (since there is no edge between vertices in S), then the cycle $\mathcal{C}$ has length at least g. The same applies if the cycle $\mathcal{C}$ goes through two new edges incident with two distinct vertices s_i in $K \cup K'$.

Case 4: The cycle $\mathcal{C}$ goes through two new edges, one of which sits inbetween $s_i \in V(H)$ and $v_{ij} \in N^2(v) \subset V(K)$, and another of which sits inbetween $u_{kij} \in V(H)$ and $v_i \in V(K)$. Since the edges $v_i v_{ij}$ have been deleted from K, we have that the cycle has length at least $g - 1 + 2 + d_H(u_{kij}, s_i) > g$.

Case 5: The cycle $\mathcal{C}$ goes through two new edges, one of which sits inbetween $s_i \in V(K)$ and $v_{ij} \in N^2(v) \subset V(K')$, and another of which sits inbetween $v_i \in V(K)$ and $v_{kij} \in V(K')$.

Now we have, by Lemma 3.1, that the cycle has length at least $d_K(v_i, s_i) + 2 + d_{K'}(v_{kij}, v_{ij}) \geq \lfloor (g-3)/2 \rfloor + 2 + g - 5 > g$, because $g \geq 11$.

Case 6: The cycle $\mathcal{C}$ goes through two new edges, one of which sits inbetween $v_i \in V(K)$ and $v_{kij} \in N(v_{ki}) \subset V(K')$, and another of which sits inbetween $v_r \in V(K)$ and $v_{krj} \in N(v_{kr}) \subset V(K')$. In this case we have that $v_i v v_r$ is a path of length 2 in K, while $d_{K'}(v_{kij}, v_{krj}) \geq g - 4$. Thus the cycle has length at least g.

Case 7: The cycle $\mathcal{C}$ goes through two new edges, one of which sits inbetween $u_{kij} \in V(H)$ and $v_{kij} \in V(K')$, and another of which sits inbetween $u_{ij} \in V(H)$ and $v_{ij} \in V(K')$. Thus the cycle has length at least $2(g-5) + 2 > g$. ∎

Remark: The purpose of the above theorem is to derive new revealing structural properties of (k, g)-cages, so that we are talking about a vertex-cut of cardinality at most $k - 1$, even though the conjecture suggests that such vertex-cut does not exist.

4 Conclusion

In this paper, we have shown that if the vertex-cut is of cardinality smaller than k or is the neighborhood of a vertex, then the graph obtained by removing this vertex-cut from a (k, g)-cage, contains exactly two components. We also have proved that if (k, g)-cages are not vertex superconnected, then the graph obtained by removing a nontrivial vertex-cut of cardinality k has at most $k - 2$ components if k is even. Therefore, we propose the following open problem.

Open Problem 1 *How many components will there be if we remove a nontrivial vertex-cut of cardinality k from a (k, g)-cage if k is odd?*

We also believe that $(4, g)$-cages are superconnected. We know that $(4, g)$-cages are 4-connected. Now we know that if a $(4, g)$-cage is not vertex superconnected then there are 2 components after deleting a nontrivial cut-set of 4 vertices. Hence we propose the following open problem.

Open Problem 2 *Are $(4, g)$-cages vertex superconnected?*

However, these problems beg further investigation and subsequent research could expand to other topics, such as to find the toughness of a cage. In this paper, we have provided some preliminary results proving that $|S|/w(G - S) \geq k/k + 1$ for any nontrivial vertex-cut S. Thus we propose the following open problem

Open Problem 3 *How many components will there be if we remove a trivial vertex-cut of cardinality greater than k from a (k, g)-cage?*

References

[1] M.C. Balbuena, A. Carmona, J. Fàbrega, M.A. Fiol, On the order and size of s-geodetic digraphs with given connectivity, *Discrete Math.* **174** (1997), 19–27.

[2] G. Brinkmann, B.D. McKay, C. Sager. The smallest cubic graphs of girth nine, *Combinatorics, Probability and Computing* **4** (1995) 317-330.

[3] G. Chartrand, L. Lesniak, *Graphs and digraphs*, Third edition, Chapman and Hall, London, 1996.

[4] M. Daven and C.A. Rodger, (k, g)-cages are 3-connected, *Discrete Math.* **199** (1999), 207-215.

[5] P. Erdos, H. Sachs, Regulare Graphen gegebener Taillenweite mit minimaler Knotenzahl, *Wiss. Z. Uni. Halle (Math. Nat.)* **12** (1963), 251–257

[6] J. Fàbrega, M. A. Fiol, Maximally connected digraphs, *J. Graph Theory* **13** (1989), 657–668.

[7] H.L. Fu, K.C. Huang, C.A. Rodger. Connectivity of cages. *J. Graph Theory* **24** (1997), 187–191.

[8] T. Jiang, D. Mubayi, Connectivity and separating sets of cages, *J. Graph Theory* **29** (1998), 35–44.

[9] Y. Lin, M. Miller and C. Balbuena, Improved lower bounds on the connectivity of $(\delta;g)$-cages, *Discrete Math.* **299** (2005), 162–171.

[10] Y. Lin, M. Miller, C. Balbuena, X. Marcote, All (k, g)-cages are edge-superconnected, *Networks* **47** (2) (2006), 102–110.

[11] X. Marcote, C. Balbuena, Edge-superconnectivity of cages, *Networks* **43** (1) (2004), 54-59.

[12] X. Marcote, C. Balbuena, I. Pelayo, J. Fàbrega, (δ, g)-cages with $g \geq 10$ are 4-connected, *Discrete Math.*, **301** no. 1 (2005), 124–136.

[13] X. Marcote, I. Pelayo, C. Balbuena, Every cubic cage is quasi 4-connected, *Discrete Math.* **266** (1-3) (2003), 311–320.

[14] T. Soneoka, H. Nakada, M. Imase, Sufficient conditions for dense graphs to be maximally connected. *Proc. ISCAS85* (1985), 811–814.

[15] T. Soneoka, H. Nakada, M. Imase, C. Peyrat, Sufficient conditions for maximally connected dense graphs, *Discrete Math.* **63** (1987), 53–66.

[16] W.T. Tutte, A family of cubical graphs. *Proc. Cambridge Philos. Soc.*, (1947), 459-474.